SPECTRAL SUBTRACTION AND SPECTRAL ESTIMATION

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ABSTRACT

The problem of spectral subtraction, to estimate the parameters of a single source in colored noise, is used to show the relationships between the likelihood formulation and spectral density estimation. Reported previously as a filter bank processing for spectral estimation, it is shown that the normalized Capon estimate is the natural tool for source location in 1-d and 2-d scenarios when the noise background estimate is faced as a spectral subtraction problem. Several simulations selecting 1-d and 2-d apertures are used to show the degree of quality achieved with the proposed formulation. Also, the Periodogram test for incoherent detection is analyzed in front of the optimum test and the herein referred to as the Capon's test.

I. INTRODUCTION

Using the maximum likelihood formulation for the problem of a line source embedded in colored noise, this work seeks the relationships between frequency detectors, spectral density estimates and spectral subtraction.

It seems clear that characterizing the source location, its power level and the spectral density of the noise, entails to estimate first the source location, second its power level and finally, by spectral subtraction, the noise spectral density. Assuming this path in the procedure, it seems also clear that a high resolution line detector is needed at the first step and this is the reason for the interest towards the spectral density estimates. For the second step, a power level estimate is required and apparently the Capon estimate has no competitor to perform such estimation. Finally, the third step reduces, again apparently, to the subtracting the estimated line contribution from the data covariance matrix. Regardless this protocol is valid in essence, there are several issues of interest, related with the maximum likelihood formulation, that preclude and arbitrary choice of the procedures used at every step.

To go through the mentioned steps, the problem of detecting a line source in colored noise has been selected. The presentation focuses on the case of an ULA array, leaving the case of 2-D apertures to the simulations section.

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The snapshot model we are assuming is formed by a single line impinging on a ULA array of Q sensors with colored noise $\underline{\omega}(n)$. It is considered that the actual complex envelop and the spatial frequency are $\alpha_e(n)$ and f_o respectively, i.e. f_o is equal to the product of the inter-element distance in wavelength by the sinus of the source elevation $d.\sin(\theta_o)$. The snapshot model is given by (1), where $\underline{S}_e$ is the (Qx1) steering vector at frequency f_o.

$$\underline{X}_n = \alpha_e(n).\underline{S}_e + \underline{\omega}_n \tag{1}$$

The log likelihood of this data snapshot is given by (2), where a_n, $\underline{S}$ and $\underline{\underline{R}}_o$ are the estimates of the complex envelope, the steering vector and the noise covariance matrix respectively.

$$\Lambda_n = Ln\left(\det\left|\underline{\underline{R}}_o^{-1}\right|\right) - \left(\underline{X}_n - a_n.\underline{S}\right)^H.\underline{\underline{R}}_o^{-1}.\left(\underline{X}_n - a_n.\underline{S}\right) \tag{2}$$

From this expression is obvious that the power level estimate is the suitable step to proceed first, instead the source location since the log-likelihood is highly non-linear on this parameter. The ML estimate of the source complex envelope is derived from the maximization of the above expression. The estimate is given by (3.a) and in (3.b) the corresponding power level estimate, where N is the number of collected snapshots.

$$a_n = \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{X}_n}{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}} \tag{3.a}$$

$$\alpha = \frac{1}{N}\sum_{n=0}^{N-1}\left|a_n\right|^2 = \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}.\underline{\underline{R}}_o^{-1}.\underline{S}}{\left(\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}\right)^2} \tag{3.b}$$

Note that regarding to (3.b) that it looks different from the traditional Capon estimate [1]. Usually it is argued that the product of the inverse of the noise covariance matrix by the steering is proportional to the vector resulting of using the data matrix instead of the noise matrix. Nevertheless, this property depends on the estimates of the noise covariance matrix and the steering of the source, which at this step are not available.

II THE LOG-LIKELIHOOD OF THE STEERING AND THE NOISE COVARIANCE

Using (3.a) in (2) and summing up for the available snapshots, the log-likelihood to be maximized is obtained.

$$\Lambda = Ln(\det(\underline{\underline{R}}_o^{-1})) - trace\left\{\underline{\underline{R}}_o^{-1}\left[\underline{\underline{R}} - \frac{\underline{S}.\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}}{\rho_o}\right]\right\} \tag{4}$$

where

$$\rho_o = \underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S} \tag{5}$$

Re-arranging terms in (4), it can also be written as (6).

$$\Lambda = Ln\left(\det\left[\underline{\underline{R}}_o^{-1}\right]\right) - trace\left[\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}\right] + \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}.\underline{\underline{R}}_o^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}} \tag{6}$$

This last formulation reveals that the optimum detection test $T_{opt}(f)$ to estimate the source location, assuming that the noise covariance is known, is a Raleigh quotient, which is close to the power estimate (3.b) derived previously. This test plays always an important role in improving detectors and beamformers [4]. It is also coherent with the white noise case since it coincides precisely with the data Periodogram.

Before going on with the procedure, it is worth studying the properties of the optimum test under perfect knowledge of the inverse noise covariance.

Assuming that the inverse of the noise covariance is known, the optimum test to locate the source steering is to maximize (7).

$$T_{opt}(f) = \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}.\underline{\underline{R}}_o^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}_o^{-1}\underline{S}} \tag{7}$$

This test is lower bounded by using the following definitions an inequality:

$$\underline{v} = \underline{\underline{R}}^{1/2}.\underline{\underline{R}}_o^{-1}.\underline{S} \;\; ; \;\; \underline{u} = \underline{\underline{R}}^{-1/2}.\underline{S} \;\; ; \;\; \|\underline{v}\|^2 \le \frac{\left|\underline{u}^H.\underline{v}\right|^2}{\|\underline{u}\|^2} \tag{8}$$

In consequence,

$$T_{opt}(f) = \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}.\underline{\underline{R}}_o^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}} \le \frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}^{-1}.\underline{S}} = T_{Cap}(f) \tag{9}$$

This reveals that, in terms of resolution the right term of (9) will be better than the optimum test, since both get the same value at the true steering. Also it is very important to remark that the so-called classic test formed by the quotient of the periodograms measures do not bound, in any way, the optimum test. Its use is only suitable in the case of white noise only; in this case the Periodogram test coincides with the optimum test, which still is bounded by the Capon test.

$$T_{Period} = \frac{\underline{S}^H.\underline{\underline{R}}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}_o.\underline{S}} \tag{10}$$

Just to put in evidence the above comments, in Figure 1, they are represented simultaneously all the tests described under the condition of perfect knowledge of the noise covariance matrix. The data are formed by a source located at the spatial frequency 0.1, the colored noise obtained from a Moving Average MA(3) model, with model coefficients (1 0 –1), and SNR equal to 8 dB. This figure reveals the claimed superiority of the Capon test over the test derived from the log-likelihood. More important is the poor performance of the Periodogram test.

III. SPECTRAL SUSTRACTION

An alternative to the direct maximization of the log-likelihood over the noise and steering parameters is to set the relationship between the noise covariance matrix and the steering to be estimated as a spectral subtraction problem. Note that noise covariance estimation is a major issue for power level estimation [2], and spectral subtraction, regardless of being heuristically in many cases, uses to be the suitable tool to perform it.

Under a spectral subtraction approach the noise covariance matrix is formulated as the subtraction of the data covariance matrix minus the source contribution .

$$\underline{\underline{R}}_{o,1} = \underline{\underline{R}} - \beta.\underline{S}.\underline{S}^H \tag{11}$$

The maximum value of β, in order to ensure that the estimated matrix is positive definite, is precisely the traditional Capon estimate. The problem is that using this estimate precludes the use of the inverse since it does not exist because the minimum eigenvalue of the estimated noise matrix is zero.

$$\beta_{max} = \frac{1}{\underline{S}^H.\underline{\underline{R}}^{-1}\underline{S}} \tag{12}$$

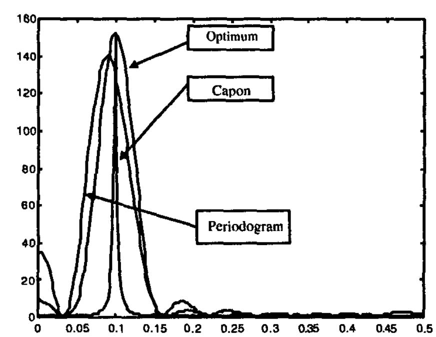

Figure 1. Optimum (-), Capon (-.-) and Periodogram test (--).

The second attempt is more suitable and faces directly the estimate of the inverse, since it is the only one that is required in the formulation of the log-likelihood when using the appropriate test for source location. This estimate is given in (13), where the second term is the rank one contribution of the vector (norm one) that nulls the contribution of the last two terms of (6). Again, the parameter γ has to be bounded in order to preserve the positive character of the estimate.

$$\underline{\underline{R}}_o^{-1} = \underline{\underline{R}}^{-1} + \gamma . \left(\frac{\underline{\underline{R}}^{-1}.\underline{S}.\underline{S}^H.\underline{\underline{R}}^{-1}}{\underline{S}^H.\underline{\underline{R}}^{-2}.\underline{S}} \right) \qquad (13)$$

This bound is given in (14), where it is clear that it is less restrictive than the case of modeling directly the noise matrix.

$$\gamma \geq -\frac{1}{\underline{S}^H.\underline{\underline{R}}^{-1}.\underline{S}} \qquad (14)$$

At the same time, the log-likelihood for this noise estimate is equal to (15), in consequence γ has to be selected in order to maximize the determinant of the inverse noise covariance matrix.

$$\Lambda = Ln(\det(\underline{\underline{R}}_o^{-1})) - (Q-1) \quad ;\forall\gamma \qquad (15)$$

Finally, with this choice for the parameter, the log-likelihood, the optimum test, described in the previous section, and the so-called normalized Capon estimate [3] are easily related as follows:

$$\Lambda + Q - 1 = Ln(T_{opt}(f)) =$$
$$= Ln\left[\frac{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{\underline{R}}.\underline{\underline{R}}_o^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}_o^{-1}.\underline{S}}\right] = Ln\left[1+\gamma\frac{\underline{S}^H.\underline{\underline{R}}^{-1}.\underline{S}}{\underline{S}^H.\underline{\underline{R}}^{-2}.\underline{S}}\right] \qquad (16)$$

In summary, viewing the problem of the noise covariance matrix estimation as a problem of spectral subtraction, carried over the inverse of the data covariance matrix, reveals that the optimum frequency detector is not the classical minimum variance beamformer. It is the so-called normalized Capon spectral estimate which provides the optimum test for frequency detection.

It may be argued that in the above formulation the parameter γ may also depend on the steering vector, in this respect, next section will describe some empirical support to the choice of this parameter, setting it to a constant value. Note that this is equivalent to set a constant value for the trace of the inverse noise covariance matrix estimate, independently of the steering selected.

Before closing this section, in Figure 2 they can be viewed: Right, the likelihood using (16); and, left, the noise spectral estimate using (13) and the Capon power level estimate.

IV SPECTRAL ESTIMATION.

Rewriting again the noise covariance estimate leaving unknown the parameter $K_o(\underline{S})$,

$$\underline{\underline{R}}_o^{-1} = \underline{\underline{R}}^{-1} + K_o(\underline{S}).\left[\underline{\underline{R}}^{-1}.\underline{S}.\underline{S}^H.\underline{\underline{R}}^{-1}\right] \qquad (17)$$

and assuming that the source is at this steering with power level α_s, the correct value of K_o is (18), where ρ is the inverse of the power level from the Capon estimate.

$$K_o(\underline{S}) = \frac{\alpha_s}{1-\rho(\underline{S})\alpha_s} \quad with \quad \rho = \underline{S}^H.\underline{\underline{R}}^{-1}.\underline{S} \qquad (18)$$

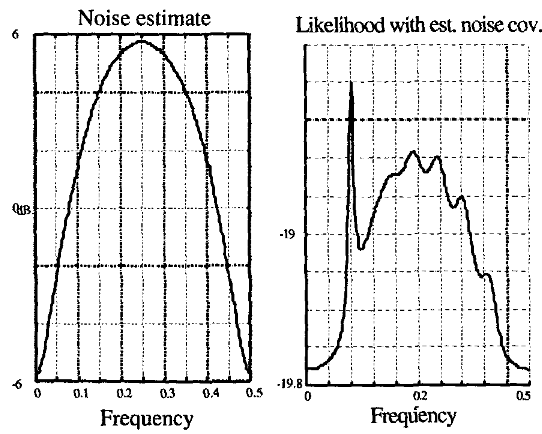

Figure 2. (Left). Capon Noise spectral density estimate. (Right). Log-Likelihood

Since the power level $1/\rho$ contains both the signal and the noise contributions and, in addition, we assume that the noise power can be formulated by a white density N_o and a shaping bandwidth $B(\underline{S})$, then

$$\frac{1}{\rho} = \alpha_s + \alpha_{on} = \alpha_s + N_o.B(\underline{S}) \qquad (19)$$

In consequence, the product $K_o(\underline{S}).\rho$ can be formulated as (20), where $\psi(\underline{S})$ is the spectral density.

$$K_o(\underline{S}).\rho = \frac{1}{N_o}\frac{\alpha_s}{B(\underline{S})} = \frac{1}{N_o}.\Psi(\underline{S}) \qquad (20)$$

Furthermore, using (20) in the corresponding log-likelihood, results in (21), which proves the efficiency and the relationship between line detectors or spectral density estimates and the maximization of the log-likelihood. At the same time, taking into account that the normalized estimate provides spectral density, ensures that the proper choice for parameter γ in the previous section is a constant independent of the steering.

$$\Lambda - Q + 1 = Ln[1 + K_o(\underline{S})\rho] = Ln\left[1+\frac{\Psi(\underline{S})}{N_o}\right] \qquad (21)$$

V. SIMULATIONS

In order to show that the framework described previously is also valid for 2-D problems, the hexagonal aperture depicted in Figure 3 has been selected for this section.

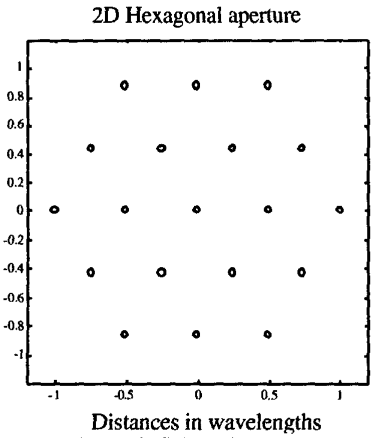

Figure 3. Selected aperture.

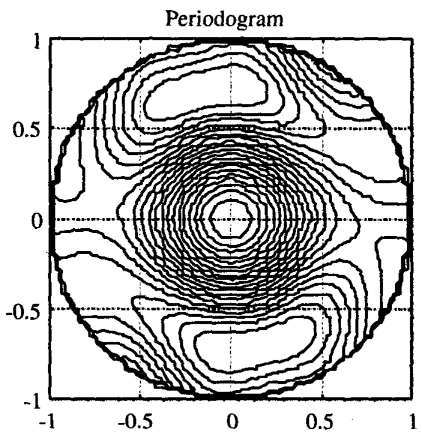

Figure 4. The noise Periodogram. Polar slowness-azimuth plot.

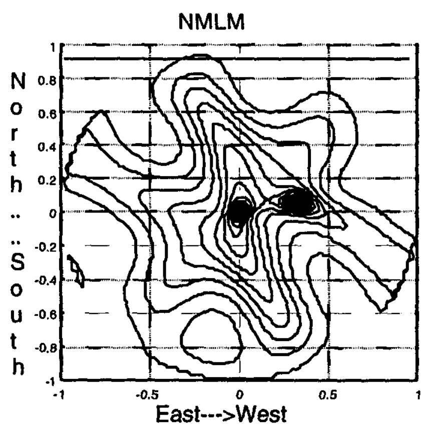

Figure 5. The Normalized estimate, proportional to the log-likelihood, for the scenario defined in the text.

The noise was spatially colored being its Periodogram estimate as depicted in Figure 4. The representation uses the slowness-azimuth plot being the south to north axis coincident with the ordinate axis of the plot.

The source was located at 20° of elevation and 80° of azimuth with a SNR of 0 dB. Figure 5 shows the normalized estimate, proportional to the likelihood. The estimated location of the source is 80.83° and 20.18°. It is important to remark that the normalized estimate (NMLM) performs like a high resolution procedure and its accurateness in the source location requires high density grid to scan for the maximum.

The Capon estimate for the power level of the background noise, with the spectral subtraction indicated previously can be viewed in Figure 6, where it is evident the similarity with the original one.

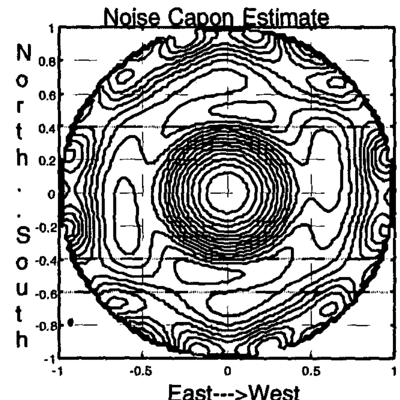

Figure 6. Capon noise power level estimated from the spectral subtraction procedure described in the text.

VI. CONCLUSIONS

It has been shown what is the relationship between spectral density estimators and source detection in colored noise. At the same time, the interest of the normalized Capon estimate has been proven to be the natural 1-d or 2-d spectral density estimate. In fact, this density estimate, for any 2-d scenarios, is superior to the available procedures of Periodogram (low resolution) and Music (high complexity and order uncertainty).

VII. REFERENCES

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